Inverse of One-to-one Functions

by CHED on March 29, 2020

lesson duration of 3 minutes
under General Mathematics

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Tags: Inverse Functions
Inverse of One-to-one Functions  (3 mins)

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Subjects: General Mathematics

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Resources

n/a

Content Standard

The learner demonstrates understanding of key concepts of inverse functions, exponential functions, and logarithmic functions.

Performance Standard

The learner is able to apply the concepts of inverse functions, exponential functions, and logarithmic functions to formulate and solve real-life problems with precision and accuracy.

Learning Competencies

The learner determines the inverse of a one-to-one function.

The learner represents an inverse function through its: (a) table of values, and (b) graph.

The learner solves problems involving inverse functions.

Motivation 1 mins

Consider the table of values for the function given by the equation $y = 2x - 1$ given below:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-4$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$-9$</td>
<td>$-7$</td>
<td>$-5$</td>
<td>$-3$</td>
<td>$-1$</td>
<td>$1$</td>
<td>$3$</td>
<td>$5$</td>
<td>$7$</td>
</tr>
</tbody>
</table>

Verify that it is a one-to-one function by showing that no two $y$-values share the same $x$-value.

Let us invert the values for $x$ and $y$: 
Does this table still represent a function?  
We should see that it can still represent a function because each $x$ value is associated with only one $y$ value.

Next consider the table of values for another function below:

\[
\begin{array}{c|cccccccc}
  x & -9 & -7 & -5 & -3 & -1 & 1 & 3 & 5 & 7 \\
  y & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

Show that the table does not represent a function because there are some $y$-values that are paired with more than one $x$-value. For example, $y = 1$ is paired with $x = 1, 2, 3, 4$.

Invert the values for $x$ and $y$. Will the resulting table still represent a function?

\[
\begin{array}{c|cccccccc}
  x & -1 & -1 & -1 & 0 & 1 & 1 & 1 & 1 \\
  y & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

The resulting table does **not** represent a function since $x = 1$ is paired with more than one $y$-value; namely, 1, 2, 3 and 4.

**‘Inverting’ functions**

The previous discussion shows that
- if the $x$- and $y$-values of a one-to-one function are interchanged, the result is a function, but
- if the $x$- and $y$-values of a function that is not one-to-one are inverted, the result is no longer a function.

**Lesson Development** 1 mins

Define the inverse of a one-to-one function.

**Definition**

Let $f$ be a one-to-one function with domain $A$ and range $B$. Then the **inverse of $f$**, denoted $f^{-1}$, is a function with domain $B$ and range $A$ defined by $f^{-1}(y) = x$ if and only if $f(x) = y$ for any $y$ in $B$.

**A function has an inverse if and only if it is one-to-one.**

As shown earlier, ‘inverting’ the $x$- and $y$-values of a function results in a function if and only if the original function is one-to-one.

**To determine the inverse of a function from its equation**

In light of the definition, the inverse of a one-to-one function can be interpreted as the same function **but in the opposite direction**, that is, it is a function from a $y$-value back to its corresponding $x$-value.
To find the inverse of a one-to-one function,

(a) write the function in the form \( y = f(x) \);

(b) interchange the \( x \) and \( y \) variables;

(c) solve for \( y \) in terms of \( x \).

This is because we are interchanging the input and output values of a function.

For the next examples, we use the definition of the inverse to verify our answers.

**EXAMPLE 1.** Find the inverse of \( f(x) = 3x + 1 \).

**Solution.** The equation of the function is \( y = 3x + 1 \). Interchange the \( x \) and \( y \) variables: \( x = 3y + 1 \)

Solve for \( y \) in terms of \( x \):

\[
\begin{align*}
  x &= 3y + 1 \\
  x - 1 &= 3y \\
  \frac{x - 1}{3} &= y \\
  y &= \frac{x - 1}{3}
\end{align*}
\]

Therefore the inverse of \( f(x) = 3x + 1 \) is \( f^{-1}(x) = \frac{x - 1}{3} \).

Ask the following questions to the class:
(a) What is the inverse of the inverse?
(b) What is \( f(f^{-1}(x)) \)? How about \( f^{-1}(f(x)) \)?

Have the class do these on the example above. Then discuss the following properties that the class should have observed from above:

**Property of an inverse of a one-to-one function**

Given a one-to-one function \( f(x) \) and its inverse \( f^{-1}(x) \). Then the following are true:
(a) The inverse of \( f^{-1}(x) \) is \( f(x) \).
(b) \( f(f^{-1}(x)) = x \) for all \( x \) in the domain of \( f^{-1} \).
(c) \( f^{-1}(f(x)) = x \) for all \( x \) in the domain of \( f \).

For the second and third properties above, it can be imagined that evaluating a function and its inverse in succession is like reversing the effect of the function. For example, the inverse of a function that multiplies 3 to a number and adds 1 is a function that subtracts 1 and then divides the result by 3.

**EXAMPLE 2.** Find the inverse of \( g(x) = x^3 - 2 \).

**Solution.** The equation of the function is \( y = x^3 - 2 \). Interchange the \( x \) and \( y \) variables: \( x = y^3 - 2 \).

Solve for \( y \) in terms of \( x \):

\[
\begin{align*}
  x &= y^3 - 2 \\
  y^3 &= x + 2 \\
  y &= \sqrt[3]{x + 2}
\end{align*}
\]
EXAMPLE 3. Find the inverse of the rational function \( f(x) = \frac{2x + 1}{3x - 4} \).

**Solution.** The equation of the function is
\[
    y = \frac{2x + 1}{3x - 4}.
\]
Interchange the \( x \) and \( y \) variables:
\[
    2y + 1 = \frac{3y - 4}{3y - 4}.
\]
Solve for \( y \) in terms of \( x \):
\[
    x = \frac{2y + 1}{3y - 4}.
\]
\[
    x(3y - 4) = 2y + 1.
\]
\[
    3xy - 4x = 2y + 1.
\]
\[
    3xy - 2y = 4x + 1.
\]
\[
    y(3x - 2) = 4x + 1.
\]
\[
    y = \frac{4x + 1}{3x - 2}.
\]

Therefore the inverse of \( f(x) \) is
\[
    f^{-1}(x) = \frac{4x + 1}{3x - 2}.
\]

EXAMPLE 4. Find the inverse of \( f(x) = x^2 + 4x - 2 \), if it exists.

**Solution.** The students should recognize that this is a quadratic function with a graph in the shape of a parabola that opens upwards. It is not a one-to-one function as it fails the horizontal line test.

*(Optional)* We can show that applying the procedure for finding the inverse to this function leads to a result which is **not** a function.

The equation of the function is \( y = x^2 + 4x - 2 \).
Interchange the $x$ and $y$ variables: $x = y^2 + 4y - 2$

Solve for $y$ in terms of $x$:

\[
x = y^2 + 4y - 2 \\
x + 2 = y^2 + 4y \\
x + 2 + 4 = y^2 + 4y + 4 \quad \text{(Complete the square.)} \\
x + 6 = (y + 2)^2 \\
\pm \sqrt{x + 6} = y + 2 \\
\pm \sqrt{x + 6} - 2 = y \implies y = \pm \sqrt{x + 6} - 2
\]

The equation $y = \pm \sqrt{x + 6} - 2$ does not represent a function because there are some $x$-values that correspond to two different $y$-values (e.g., if $x = 3$, $y$ can be 1 or $-5$). Therefore the function $f(x) = x^2 + 4x - 2$ has no inverse function.

**EXAMPLE 5.** Find the inverse of $f(x) = |3x|$, if it exists.

**Solution.** Recall that the graph of $y = |3x|$ is shaped like a "V" whose vertex is located at the origin.

This function fails the horizontal line test and therefore has no inverse.

**Alternate Solution.** We can also show that $f^{-1}$ does not exist by showing that $f$ is not one-to-one. Note that $f(1) = f(-1) = 3$. Since the $x$-values 1 and $-1$ are paired to the same $y$-value, then $f$ is not one-to-one and it cannot have an inverse.

(Optional) Again, if we apply the procedure for finding the inverse of a one-to-one function, a problem occurs:

The equation of the function is $y = |4x|$.

Interchange $x$ and $y$: $x = |4y|$.

Solve for $y$ in terms of $x$:

\[
x = |4y| \\
x = \sqrt{(4y)^2} \quad \text{(Recall the definition $|x| = \sqrt{x^2}$)} \\
x^2 = 4y^2 \\
\frac{x^2}{4} = y^2 \\
\pm \sqrt{\frac{x^2}{4}} = y \implies y = \pm \frac{x}{2}
\]

Here, $x = 2$ will correspond to $y = 1$ and $y = -1$, so $y = \pm \frac{x}{2}$ is not a function. Therefore $f(x) = |3x|$ has no inverse function.
EXAMPLE 6. To convert from degrees Fahrenheit to Kelvin, the function is 
\[ k(t) = \frac{5}{9} (t - 32) + 273.15 \], where \( t \) is the temperature in Fahrenheit (Kelvin is the SI unit of temperature). Find the inverse function converting the temperature in Kelvin to degrees Fahrenheit.

**Solution.** The equation of the function is 
\[ k(t) = \frac{5}{9} (t - 32) + 273.15 \],

To maintain \( k \) and \( t \) as the respective temperatures in Kelvin and Fahrenheit (and lessen confusion), let us not interchange the variables. We just solve for \( t \) in terms of \( k \):

\[
\begin{align*}
  k &= \frac{5}{9} (t - 32) + 273.15 \\
  k - 273.15 &= \frac{5}{9} (t - 32) \\
  \frac{9}{5} (k - 273.15) &= t - 32 \\
  \frac{9}{5} (k - 273.15) + 32 &= t 
\end{align*}
\]

Therefore the inverse function is 
\[ t(k) = \frac{9}{5} (k - 273.15) \] where \( k \) is the temperature in Kelvin.

**Seatwork** 1 mins

The following activities can be given as either homework or seatwork.

**Seatwork 1.** Give 3 examples of situations that can be represented as a one-to-one function and two examples of situations that are not one-to-one.

**Sample Answer:** Vehicles to plate numbers, movie tickets to seat numbers, presidents or prime ministers to countries, mayors to cities or towns

**Seatwork 2.** Choose a situation or scenario that can be represented as a one-to-one function and explain why it is important that the function in that scenario is one-to-one.

**Sample Answer:** A person must have only one tax identification number (TIN) so that all the taxes he pays can be accurately recorded. If he has two TINs, the BIR might think that he did not pay all his taxes if his payments are split between multiple TINs. If a single TIN has two persons associated to it, then it would not be possible to ascertain which person is paying proper taxes and which is not.

**Seatwork 3.** Find the inverse functions of the following one-to-one functions:
(a) \( f(x) = \frac{1}{2}x + 4 \)
Answer: \( f^{-1}(x) = 2x - 8 \)
(b) \( f(x) = (x + 3)^3 \)
Answer: \( f^{-1}(x) = \sqrt[3]{x} - 3 \)
(c) \( f(x) = \frac{3}{x - 4} \)
Answer: \( f^{-1}(x) = \frac{4x + 3}{x} \)
(d) \( f(x) = \frac{x + 3}{x - 3} \)
Answer: \( f^{-1}(x) = \frac{3x + 3}{x - 1} \)
(e) \( f(x) = \frac{2x + 1}{4x - 1} \)
Answer: \( f^{-1}(x) = \frac{x + 1}{4x - 2} \)

**Seatwork 4.** Show that \( f(x) = |x - 1| \) is **not** a one-to-one function.

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